Metric degeneracy: why relativistic gravity is less susceptible to underdetermination and separatism than Newton-Cartan theory

Ruward Mulder, University of California in Irvine
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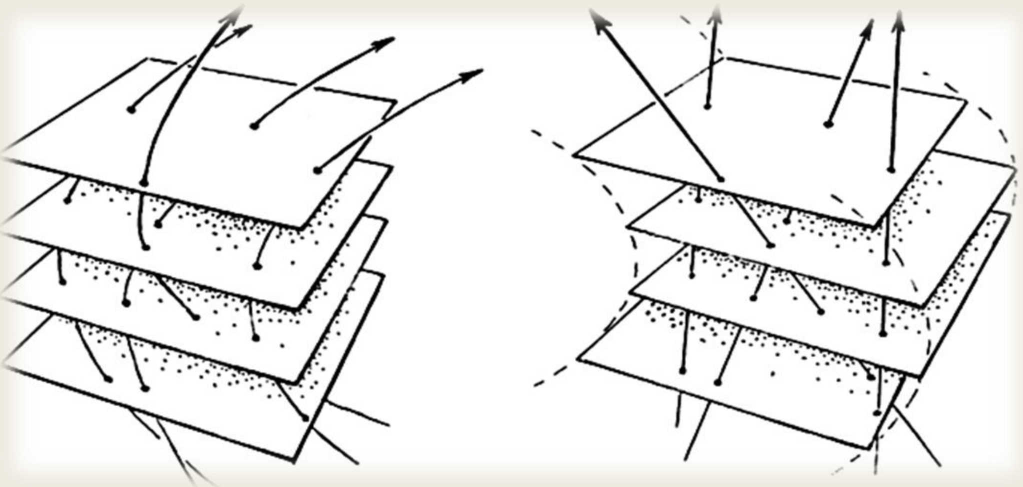

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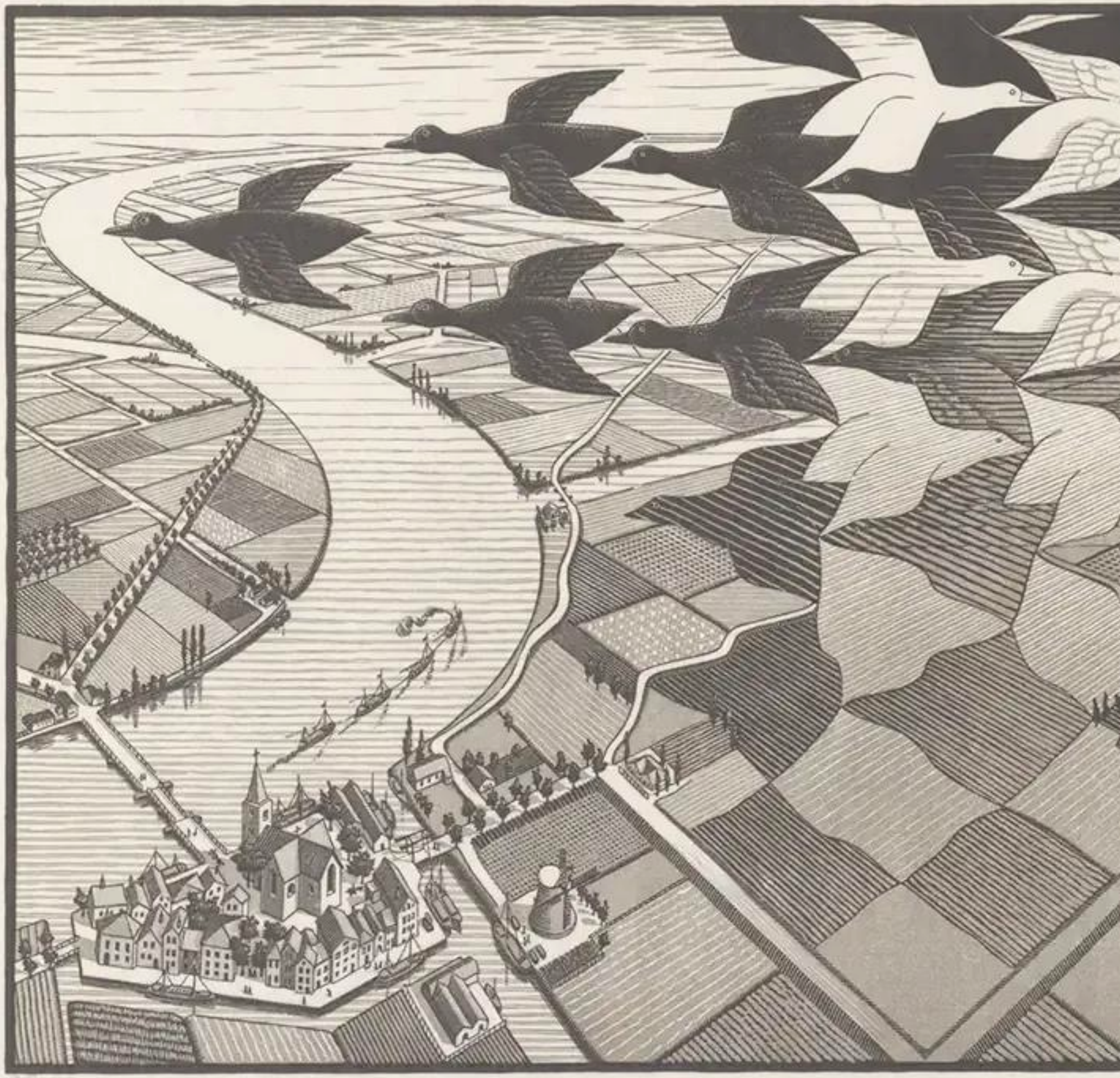


Synopsis

1) Two proofs (Weatherall & Manchak's 2014) show that:

→ non-relativistic gravity is more susceptible than general relativity to conventionalism through "universal forces" (in some Reichenbachean sense).

But: not so obvious why this discrepency arises.

2) A recently clarified metaphysical view "unitism" and its anti-thesis "separatism" (Gilmore, Costa, Calosi 2016).

→ are space and time just different kinds of spacetime or are they separate entities?

But: not obvious what mathematical underpinnings support unitism / separatism.

Claim: *Metric degeneracy* is the crucial disanalogy between relativistic and non-relativistic spacetime theories that underlies underdetermination and support for separatism of Newton-Cartan theory.

Geometric conventionalism through universal effects

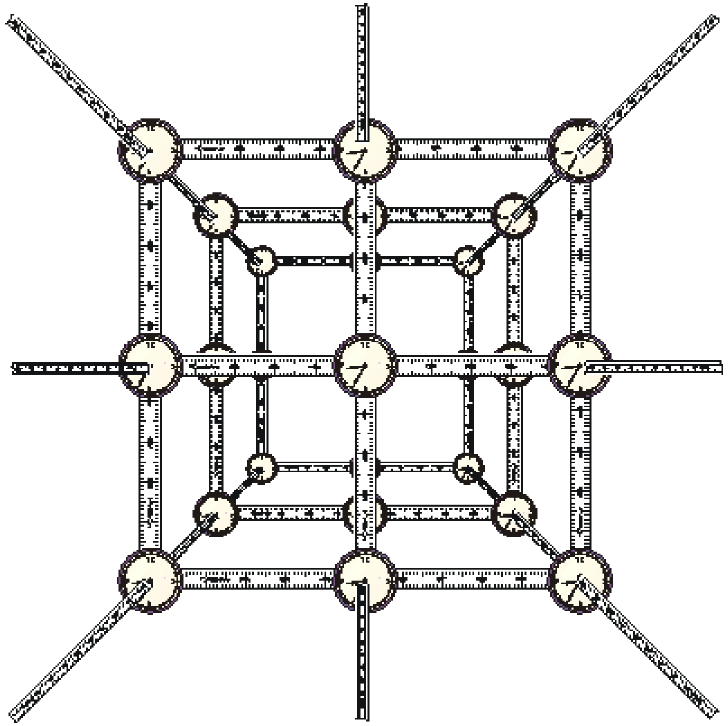
Universal forces and (non-) relativistic gravity theories

Metric degeneracy and Newton-Cartan theory

Unity, unitism, unification

Universal “forces”: 100 years of trading off geometries (1/3)

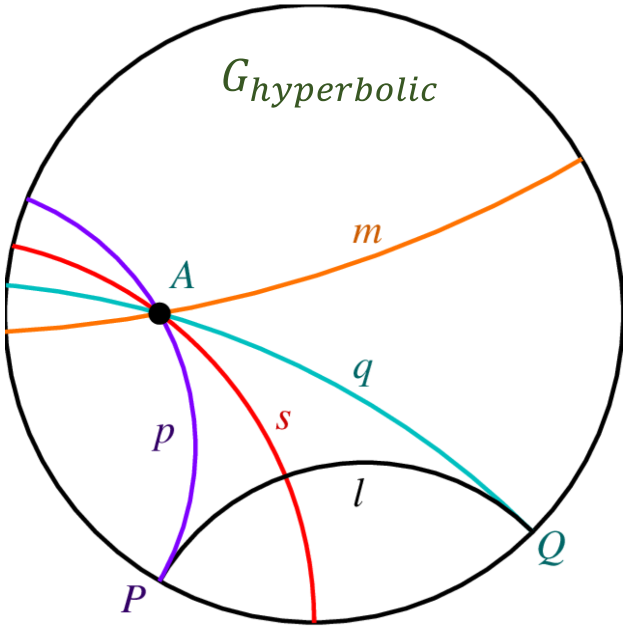
Physical geometry (Helmholtz 1866): empirical question for the *physical* geometry of the world: rods and clocks.



Universal “forces”: 100 years of trading off geometries (2/3)

Physical geometry (Helmholtz 1866): empirical question for the *physical geometry* of the world: rods and clocks.

Geometric holism based on Poincaré’s equivalent proofs (1891): there are many empirically equivalent combinations of geometries [G] and “**universal forces**” (“universal effects”) [F]: {G+F , G’+F’ , G’’+F’’, ... }.



Universal "forces": 100 years of trading off geometries (3/3)

Physical geometry (Helmholtz 1866): empirical question for the *physical geometry* of the world: rods and clocks.

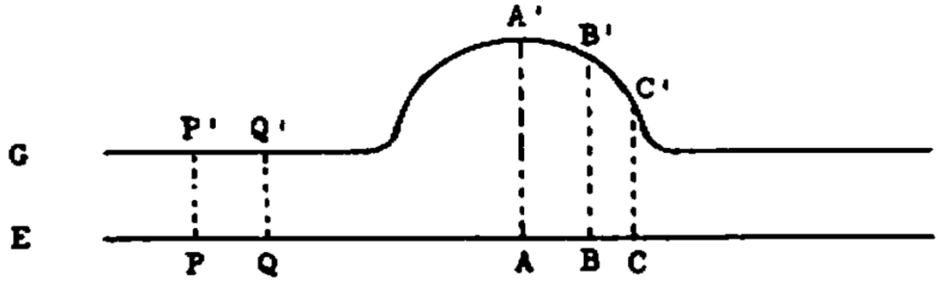
Geometric holism based on Poincaré's equivalent proofs (1891): there are many empirically equivalent combinations of geometries [G] and "**universal forces**" ("universal effects") [F]: {G+F , G'+F' , G''+F'', ... }.

Conventionalism about space(time): ascertaining the physical geometry requires (in some way) a conventional choice.
→ Loosely associated with Poincaré, Duhem, Schlick, Carnap, and others, and Reichenbach (1926, Sec. 8).:

Theorem θ: "Given a geometry *G'* to which the measuring instruments conform, we can imagine a universal force *F* which affects the instruments in such a way that the actual geometry is an arbitrary geometry *G*, while the observed deviation from *G* is due to a universal deformation of the measuring instruments."[1]

Theorem *θ* shows all geometries to be equivalent; it formulates the *principle of the relativity of geometry*. It follows that it is meaningless to speak about one geometry as the *true* geometry.

An effect (force) is universal iff it
1. cannot be screened off by insulating walls
2. acts equally on all materials/particle species
Otherwise it is a differential effect (or force)



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Weatherall & Manchak (2014): “The Geometry of Conventionality”

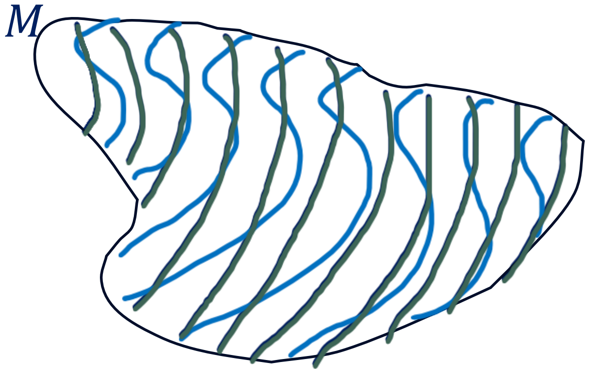
Can these trade-offs between geometry and universal forces *always* be made given the mathematics of modern spacetime theories?

Weatherall & Manchak prove that – given a force field in the “standard sense” (i.e., acting on a test body in the geodesic equation like a rank-2 tensor):
→ **Proposition 1. (Geometrised) Newtonian gravity**: yes, this is always possible.
→ **Proposition 2. General relativity**: no, this is not always possible.

Striking:
(i) Much-needed **rigour** to a conceptual debate;
(ii) Explicit **theory-dependence**;

However, there is enough to clarify:
(iii) Are the **assumptions** justified? (Dürr & Ben-Menahem 2022, Mulder 202?a)
(iv) Little engagement with **Reichenbach’s project**;
(v) No explanation / discussion of the **discrepancy**!

Spacetime (M, g, ∇)
→ geodesics: $\xi^b \nabla_b \xi^a = 0$



Spacetime $(M, \tilde{g}, \tilde{\nabla})$
→ geodesics: $\tilde{\xi}^b \tilde{\nabla}_b \tilde{\xi}^a = 0$

Spacetime $(M, \tilde{g}, \tilde{\nabla})$
→ dynamics: $\tilde{\xi}^b \tilde{\nabla}_b \tilde{\xi}^a + F^a{}_b \tilde{\xi}^b = 0$

Proposition 1: Newtonian gravity is underdetermined

<u>The non-relativistic case</u> (Prop. 1.)
Fix a classical spacetime (M, t_a, h^{ab}, ∇) and consider an arbitrary torsion-free derivative operator on M, $\tilde{\nabla}$, which we assume to be compatible with t_a and h^{ab}. Then there exists a unique antisymmetric field F^{ab} such that given any timelike curve γ with unit tangent vector field ξ^a, $\xi^n \nabla_n \xi^a = 0$ iff $\xi^n \tilde{\nabla}_n \xi^a = F^a{}_n \xi^n$, where $F^a{}_n = h^{am} F_{mn} \xi^n$. (W&M, p. 240)

Newton-Cartan theory has (pseudo-)metrics t_a and h^{ab}.

We look at the difference tensor $C^a{}_{bc}$ defined between two connections: $\nabla = (\tilde{\nabla}, C)$.

(Malament 2012, Prop (4.1.3): If ∇ is compatible with both ($\nabla_b\, t_a = 0 = \nabla_b h^{ab}$) then $\tilde{\nabla}$ is compatible with t_a and h^{ab} iff C is of the form $C^a{}_{bc} = 2h^{an} t_{(b} \kappa_{c)n}$, for κ a smooth anti-symmetric field.

Then compute the acceleration in the new geometry (on a ∇-geodesic of the old geometry):

$$\tilde{a}^a = \xi^b \tilde{\nabla}_b \xi^a = -\, C^a{}_{nm} \xi^n \xi^m = -\, 2h^{ar} t_{(n} \kappa_{m)r} \xi^n \xi^m = -2h^{ar}\, F_{mr}\, \xi^m,$$

for $F_{mr} = t_{(n} \kappa_{m)r}$ making good geometric sense (e.g., acts similarly to the Faraday tensor)

Proposition 2: GR is less susceptible to underdetermination

<u>The relativistic case</u> (Prop. 2.)

Let (M, g_{ab}) be a relativistic spacetime, let $\tilde{g}_{ab} = \Omega^2 g_{ab}$ be a metric conformally equivalent to g_{ab} and let ∇ and $\tilde{\nabla}$ be the Levi-Civita derivative operators compatible with g_{ab} and $\tilde{g}_{ab}$, respectively. Suppose Ω is nonconstant. Then there is no tensor field F_{ab} such that an arbitrary curve γ is a geodesic relative to ∇ if and only if its acceleration relative to $\tilde{\nabla}$ is given by $F^a{}_n \tilde{\xi}^n$, where $\tilde{\xi}^n$ is the tangent field to γ with unit length relative to $\tilde{g}_{ab}$. (W&M, p. 242-3)

Compute the difference tensor C (for $\nabla = (\tilde{\nabla}, C)$) for *conformally equivalent* spacetimes ($\tilde{g}_{bc} = \Omega^2 g_{bc}$):

$$C^a{}_{bc} = \frac{1}{2} g^{an}(\nabla_n g_{bc} - \nabla_b g_{nc} - \nabla_c g_{bn}) = \frac{1}{2\Omega^2}(g_{bc} g^{an} \nabla_n \Omega^2 - \delta_c{}^a \nabla_b \Omega^2 - \delta_b{}^a \nabla_c \Omega^2).$$

With normalization $g_{ab}\, \xi^a \xi^b = 1 = \tilde{g}_{ab} \tilde{\xi}^a \tilde{\xi}^b = g_{ab} \Omega^2 \tilde{\xi}^a \tilde{\xi}^b \rightarrow \tilde{\xi}^a = \Omega^{-1} \xi^a$.

Geodesic equation $\tilde{\xi}^b \tilde{\nabla}_b \tilde{\xi}^a = \tilde{\xi}^b \nabla_b \tilde{\xi}^a + C^a_{bc} \tilde{\xi}^a \tilde{\xi}^b = \cdots = \cdots = \frac{1}{\Omega^3}(\xi^b \xi^c - g^{an}) \nabla_n \Omega$.

If F_{ab} exists, then (on a ∇-geodesic): $\tilde{\xi}^b \tilde{\nabla}_b \tilde{\xi}^a = F^a_m \tilde{\xi}^m = \frac{1}{\Omega} \tilde{g}^{an} F_{nm} \xi^m = \frac{1}{\Omega^3}(\xi^b \xi^c - g^{an}) \nabla_n \Omega$.

→ This leads to a contradiction

→ Thus no F_{ab} exists for GR!

→ But W&M don't dwell at length on why this difference arises.

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Metric *non*-degeneracy

A semi-Riemannian metric on manifold M is a smooth symmetric rank-2 tensor field g_{ab} that is **invertible** :

(1) There exists a tensor field g^{bc} on M such that $g_{ab}g^{bc} = \delta^c_a$.

If this holds then g_{ab} is called *non-degenerate*.
→ this is a **local notion**: a metric is degenerate at point $p \in M$ if the bilinear $g_p: T_pM \times T_pM \to \mathbb{R}$ is degenerate.

But this is **equivalent** to the statement that (see Malament 2012, p.74):

(2) For all p in M, and all vectors at p, if $g_{ab}\xi^a = \mathbf{0}$, then $\xi^a = \mathbf{0}$.

Thus, the contraction map $g_{ab}\xi^a \mapsto \xi^a$ is **injective**: the only vector send to zero is the zero vector.

→ Every vector $\xi^a \in T_pM$ can be uniquely associated with a covector $\omega_a \in T^*_pM$.

→ This is not the case in Newton-Cartan theory.

Metric degeneracy: why there exist an F_{ab} for NCT but not for GR

In Newton-Cartan theory the (pseudo-)metrics t_a, h^{ab} are *degenerate*: there *are* non-zero vectors that get assigned zero "length" in T_pM:

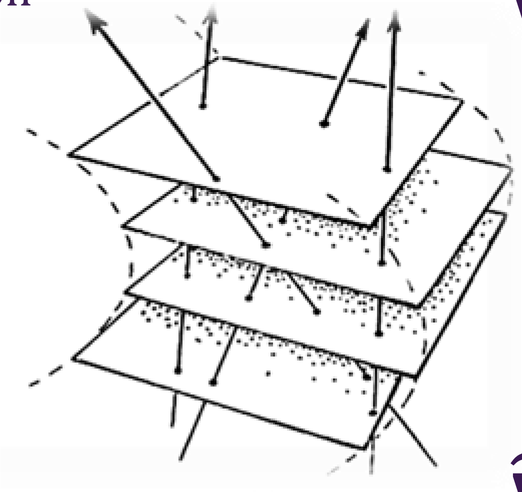
- h^{ab} annihilates anything temporal: it can project onto space, but it cannot give inner products of arbitrary vectors.
- range of h^{ab} lies entirely in the subspace of the tangent space orthogonal to t_a. That is $h^{ab}t_a = 0$.
- t_a defines a preferred direction in spacetime.

- Other than in GR, there is *no unique* connection compatible with t_a and h^{ab}:
  $\nabla t = \nabla h = 0 \; and \; \tilde{\nabla} t = \tilde{\nabla} h = 0 \; . \quad Yet: \tilde{\nabla} \neq \nabla$
- Spatial relations within each leaf are still agreed upon.

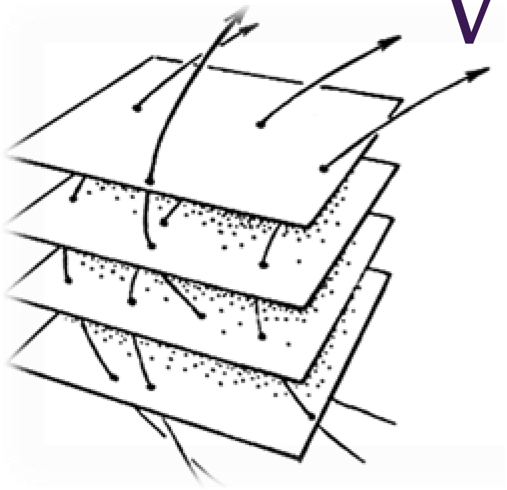
Thus, underdetermination arises as multiple correct ways to link the leafs through **motion**, namely (By Prop (1)): $a^a = \xi^b \nabla_b \xi^a = 0$ or $\tilde{a}^a = \xi^b \tilde{\nabla}_b \xi^a = -2h^{ar} F_{mr} \xi^m$.

In GR, the metric uniquely determines the connection: any change in connection requires a change in the metric
→ this is why it cannot be captured by a standard force field F_{ab}.

∇



$\tilde{\nabla}$



Conventionalism, incompleteness, and the Kaluza-Klein miracle

Roberts (202?) argues that conventionalism is a consequences of **incompleteness.**

Main example. Through the **Nash embedding theorem** one can take any manifold and represent it as a smooth surface inside a higher-dimensional flat space (isometrically).

- Intrinsic curvature is turned into extrinsic curvature via embedding functions:
- Take a GR model (M, g) and embed it isometrically into a 5D flat space $(\mathbb{R}^5, G_{AB})$ with embedding map $\phi: M \hookrightarrow \mathbb{R}^5$ such that $g_{ab}(x) = \partial_a \phi^A(x) \partial_b \phi^B(x) G_{AB}$.
- Extrinsic curvature is not uniquely fixed by the intrinsic geometry.

→ **Dimensional conventionality**: any geometry we experience could be viewed as the result of embedding into a higher-dimensional flat space!

→ **Embedding map conventionality**: Nash's theorem guarantees existence of ϕ, not uniqueness: typically there are many ϕ‘s per g.

The “**Kaluza-Klein miracle**”: unification through compactification 5th dimension: both GR and EM emerge from a single geometric structure that is physically motivated and is such that the choice of geometry is no longer arbitrary.

Does that mean **NCT incomplete** in Roberts’ sense? I think so…
→ It does not uniquely determine the affine structure: **multiple empirically indistinguishable but geometrically distinct models**, which (in Roberts’ sense) leaves too much room for conventions and needs to be “completed”.

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Unitism and Separatism

Inspired by Minkowski:

"Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality"

Gilmore, Costa, Calosi (2016) observe there are ***three*** **four-dimensionalisms**:

1. B-theory (of time) — as opposed to non-B-theory (like A-theory)
2. Perdurantism — as opposed to Endurantism
3. Unitism — as opposed to Separatism

The idea is to ask whether space and time are "unified" into spacetime or maintain "separate" metaphysical distinctness.

Are 'points in space' & 'instances of time' are just sorts of 'spacetime events'?

How to make that more precise?

Formulating Unitism and Separatism

<u>Several clarifications of **unitism**</u>:

1. ‘Points in space’ & ‘instances of time’ are just sorts of ‘spacetime events’;
2. Spatiotemporal distances are invariant, while spatial and temporal distances are not;
3. Nothing can exist at a time while not existing at a space, or *vice versa;*
4. Space and time are unified in a 4D manifold. (not specified in what way);
5. Spacetime points are ‘simples’, more fundamental than complex entities space and time.

<u>And of **separatism**</u>:

1. Space and time do not have shared parts;
2. Space *persists* through time (i.e., denial of perdurantism about space);
3. Space is entirely made of spatial points.

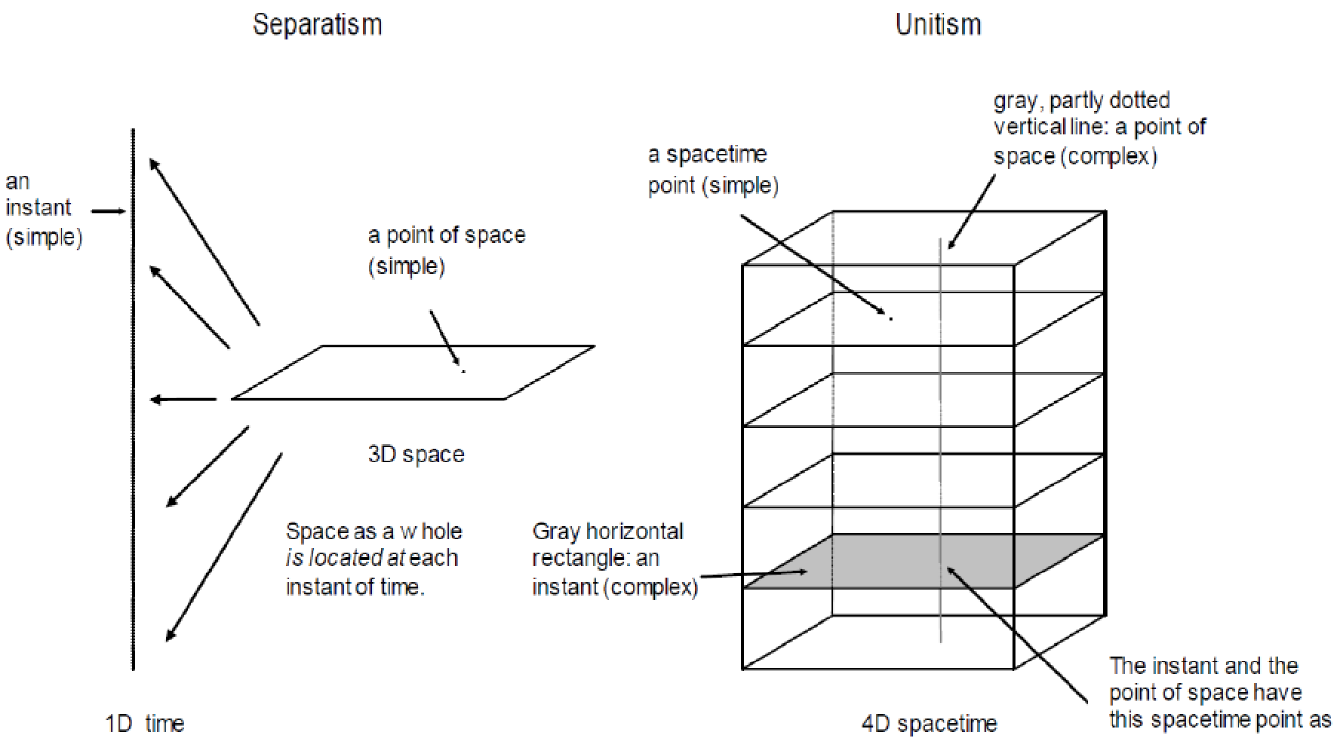

Figure 1

Metric degeneracy of Newton-Cartan is what supports Separatism

Metric degeneracy by itself supports Separatism:

- For NCT, t_a and h^{ab} naturally mirror a separatist ontology:
  - there is no single unified metric uniting space and time into a unique 4D geometric structure
  - space and time are primitive “simples”
  - nor are there invariant 4D intervals (like $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$)
  - in principle, it makes semantical sense to exist in time but not in space.
  - NCT is incomplete: there are conventional choices to be made how to attach a spatial slice at each time instant.

- In general, degeneracy by itself is a sufficient condition to support Separatism:
  → If the kernel contains not just the zero vector, then some non-zero vectors that get assigned zero length.
  → Space and time cannot be attached to each other in a **unique** way!

NCT in Separatist terms **(NCT-I)**:

- Time evolves and space evolves through time.
- **Dynamics** describes temporal evolution of spatial configurations.

NCT in Unitist terms? (**NCT-II**):

- **Demand** that the split into “space” and “time” is not absolute: all geometry is encoded in a full 4D structure.
- Requires extra conditions: a unique (global) connection ∇ determined by **mass distribution + Trautman conditions**.
- Dynamics is just geometric structure constraining worldlines.

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