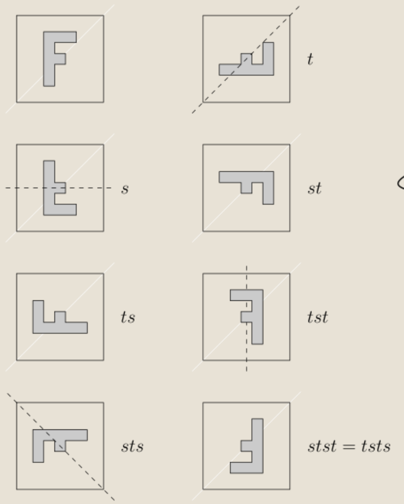
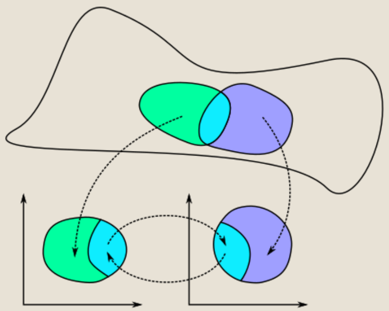
1926 – 2026: One hundred years of geometric objects

Order of presentation:

Ruward Mulder
Eleanor March
Silvester Borsboom
James Read
John Dougherty





$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

How to be a good geometer:

Concepts of "geometric object" through the eyes of Jan Arnoldus Schouten

Ruward Mulder
University of California, Irvine

Congrès international des mathématiciens, Oslo 1936 (Oslo National Library; Hollings & Siegmund-Schultze 2020)

Part 1: The history and philosophy of the evolving concept of “geometric object”

Part 2: Concepts of “geometric object” through the eyes of Dutch mathematician Jan Arnoldus Schouten

So what is a geometric object?

"Since the beginning of this century attempts have been made to define a concept general enough to include all structures which appear meaningfully in local differential geometry. Tensors were not general enough, as connections, very popular since their discovery in 1918, were not tensors. The same applies to bundles of jets and many other more recent constructions."

– Sarah Salvioli (1972) "On the theory of geometric objects"

Fields are everywhere: tensors, connections, differential invariants, gauge fields, spinors, pseudotensors, etc. ...

→ these fields are supposed to **survive coordinate redescriptions**

→ yet an elucidating concept has *largely disappeared from philosophical view*: **the geometric object**

→ The geometric-objects programme made **components** and their **transformation rules** the explicit object of study.

→ the geometric object concept explains how **different coordinate representations** belong to **the same object**.

Field: A rule assigning to each point $p \in M$ a value $F(p)$ of some specified kind: scalar, tensor, connection, ...
Coordinate system: A smooth chart $x\colon U \subseteq M \to \mathbb{R}^n$, assigning each $p \in U$ an n-tuple $x(p) = (x^1, (p), \ldots x^n(p))$.
Components: numerical functions representing a field relative to a coordinate system and its. E.g. vector $V = V^\mu \frac{\partial}{\partial x^\mu}$.

A **geometric object** has:

1. components in each coordinate system;
2. a **transformation rule** for translating those components into another coordinate system;
3. a **consistency condition**: for coordinate systems O, O', O'', we have $O \to O'' = O \to O' \to O''$.

→ **Surprising, the historical roots and conceptual foundations of the geometric objects are quite ill-understood!**

Geometric objects: why revive the tradition?

We believe there has been a tangible case of "Kuhn loss": modern debates could be better if they were informed by the hisotry of the concept of geoemtric object:

Kleinian methods: Can geometry be recovered from transformations and invariants? *(Wallace 2019; Barrett & Manchak 2024, 2026; Gomes et al. 2024; March 2025; cf. Schouten 1926)*

Gravitational stress-energy: Does gravity possess a meaningful local energy density? *(Pitts 2010; Curiel 2019; Duerr 2021; Read 2022):* $\partial_\mu \left[\sqrt{-g}\left(T^{\mu}{}_{\nu} + t^{\mu}{}_{\nu}\right)\right] = 0$, where $t^{\mu}{}_{\nu}$ is the Einstein pseudotensor. (think also about)

General covariance: discussions of general covariance -- merely formal coordinate freedom or structural property of a sapcetime theory? (Dewar 2020; March & Weatherall 2026; Weatherall and March 2026).

Pseudotensors / non-geometric objects: Components in every chart but without coherent general transformation law. *(Pitts 2010; Read 2022; cf. Nijenhuis 1952).*

Noether symmetries and conservation laws: What changes when global symmetry becomes local gauge symmetry? *(See Borsboom, this symposium; cf. Noether 1918)*

Gauge fields and natural bundles and Spinors: Do we need restricted coordinates, nonlinear transformations, or a broader framework? *(Pitts 2012; cf. Kolář et al. 1993; Dougherty, this symposium)*

The Padua school

Schouten school: the geometric objects programme (1926-1952)

Cartan

1926
Localizing Erlangen
Schouten: connections exceed Klein's scheme; Veblen & Thomas define invariants by components and transformation law.

1932/35
Expansive object theory
Veblen & Whitehead and Schouten & van Dantzig: points, hypersurfaces, curves, fields; objects transform "into themselves".

1952
Nijenhuis
Mature theory: components in every chart + induced transformations; pseudogroups, equivalence, deformation.

Erlangen + Göttingen

1929/30
invariants and objects
Veblen: geometry as theory of an invariant; Schouten & van Kampen prefer "geometric object".

1936/37
Pointwise turn (March 2025)
Oslo conference + Schouten & Haantjes: finite numerical components at each point; local fields become central.

The Princeton School

Part 1: The history and philosophy of the evolving concept of "geometric object"

Part 2: Concepts of "geometric object" through the eyes of Dutch mathematician Jan Arnoldus Schouten

Jan Arnoldus Schouten 1883–1971

Youth in Nijmegen; fluency in both Dutch and German.

- 1901-1912: Electrical engineering at Delft; worked at Siemens in Berlin; and played significant role in the electrification of Rotterdam.
- 1912 inheritance made study of mathematics possible.
  → Graduates in 1914 at Delft, PhD on tensor analysis: *Grundlagen der Vektor- und Affinoranalysis*
- 1914: Professorship at Delft, where he would stay for almost 30 years.
- 1917: simultaneously discovers parallel transport (with Levi-Civita).

- *Negotiating a continuous assistant position* allows him to build the “**Schouten school**”:
  → Johanna Manders; → Dirk-Jan Struik; → David van Dantzig; → Johannes Haantjes;
  → Egbert van Kampen; → Wouter van der Kulk; → and students, incl. Albert Nijenhuis.

Beyond his output, Schouten…

- …was pivotal for *applied* mathematics in The Netherlands;
- …founded the Mathematical Centre at Amsterdam (in 1946);
- …pushed the advance of the careers of his assisants & students;
- …lamented the treatment of women in maths (Manders*, Noether, Ruth Struik)
- …carefully managed various editions of his papers, made copies of letters sent, and kept an organised archive;
- …left this archive to the CWI (Centrum Wiskunde en Informative) in Amsterdam.

*Johanna Manders had graduated her phd *cum laude*, but there was no place for her – the first female reader in Delft was appointed in 1947.
→ Patent Office; → nuclear physics expert; → argued (with some success) that girls in secondary education should receive instruction in differential and integral calculus.

The abstractization of geometry

Der Ricci Kalkül (1924) is fully within Kleinian school:

- Schouten realizes tensors are not enough: densities and Christoffel symbols need wider category:
  → Geometric Object = components + coherent transformation
- classifies all coordinate expressions by transformation behaviour.

But in his own notation – a “direct” vector-affinor formalism with auxiliary “**ideal symbols**” $a_{\mu\lambda} \coloneqq \partial a^{\mu}/\partial a^{\lambda}$, $g_{\mu\sigma} " = " a_{\mu} a_{\sigma} " = " b_{\mu} b_{\sigma}$, so that $\frac{\partial g_{\mu\nu}}{\partial x^{\nu}} = a_{\mu\nu} a_{\sigma} + a_{\mu} a_{\sigma\nu}$.

Also, e.g. a fourth-order curvature tensor was represented by $K^{\cdot\cdot\cdot\cdot}$

→ this notation brings him some notoriety.

Adriaan Brouwer: “Isn’t that the guy who translated geometry into jibberish *[hottentots]*” (cf. Alberts 1998)

Hermann Weyl speaks of “Orgien des Formalismus” (cf. Alberts 1998)

Schouten later reflects: I could strangle *[erdrosseln]* its author (paraphrase of Struik 1989)

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- Schouten finds a way to improve on notation
  → Correspondence with Ricci: index discipline (cf. Ricci 1926)
  → Cartan collaboration confronted showed notational clash with Cartan's exterior differentiation and differential forms (Schouten&Cartan 1926a,b; Cogliati& Mastrolia 2018; Schouten 1954)

- **Abstractization**: replacing coordinate arrays and componentwise transformations with coordinate-free invariant objects

→ However, Schouten would regularly lament this abstractication of geometry: he cherished his pseudotensorial constructions.

Adriaan Bro
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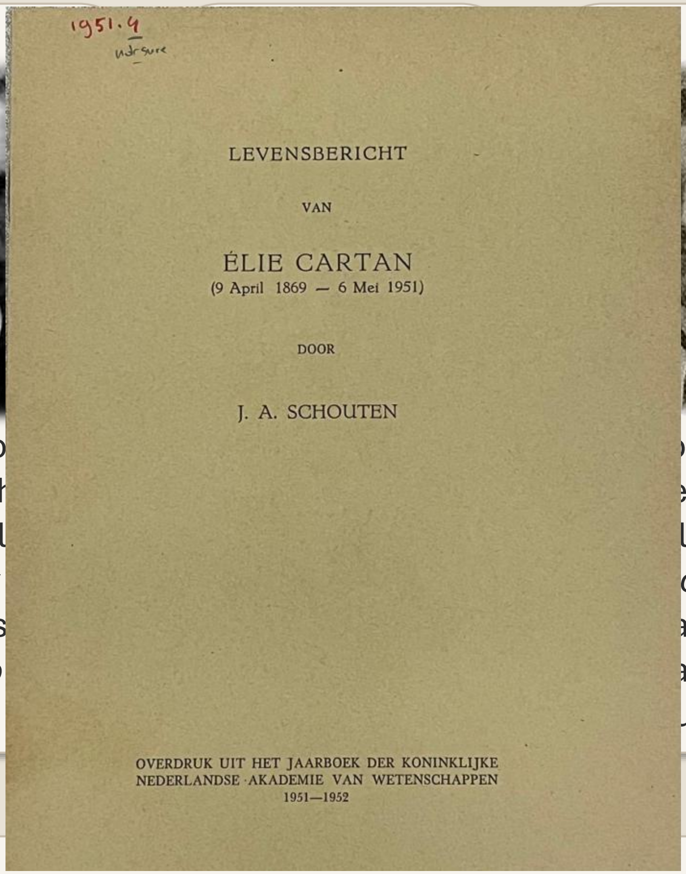

1951.4
LEVENSBERICHT
VAN
ÉLIE CARTAN
(9 April 1869 — 6 Mei 1951)
DOOR
J. A. SCHOUTEN

OVERDRUK UIT HET JAARBOEK DER KONINKLIJKE NEDERLANDSE AKADEMIE VAN WETENSCHAPPEN
1951—1952

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Cartan's method of "Pfaffian forms and his own notation, which had already been developed in his dissertation of 1894 [..] has in his hands developed into an instrument whose operation sometimes appears magical. In this respect he has been equalled by none of his pupils; one may very well learn his method, but not his manner of working, which is totally unique and only possible for a mathematician such as he, who knows his entire domain so intimately and apprehends it so intuitively."

– Schouten (1951), biographical notice of Élie Cartan (9 April 1869 – 6 May 1951)

How to be a (good) geometer?: “Het emotioneele element” in Oslo 1936

Index-free notations solved problems but also created **an identity crisis.**

→ Schouten & Van Dantzig (1935): Geometry has no timeless, fixed definition; it must be defined *a posteriori* from contemporary practice.

→ Is the content of geometry exhausted by transformation behaviour, by that which survives coordinate redescription?

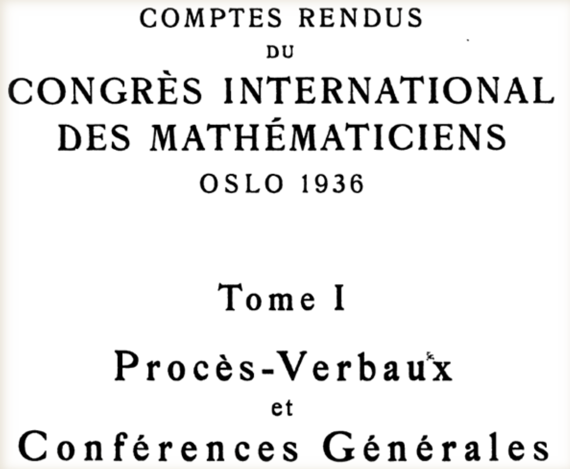
Oslo 1936 :

- Cartan delivers major plenary lecture (which Schouten chaired!): he **anounces the end of the geoemtric objects programme**: emphasis should be on moving frames, equivalence, and differential forms.
- Schouten: invariant symbolism “masks completely what there is in it *that is geometrical in the intuitive sense of the word*.”
- Schouten & Haantjes’ “On the theory of the geometric object” -- published (1937), but communicated before Oslo – formulate geometric objects to local fields with pointwise numerical components (cf. March 2025).
  → Onwards to the modern geometric object concept.

In two Dutch speeches (1939, 1949), Schouten describes the dispute as driven by **the** “emotional element” → what geometers feel to be genuinely geometric.

- invariant symbolism brought “ease and saving of labour” but could resemble “**secret scripts**” and make neighbouring fields harder to enter.
- The “emotional element” is the historically shifting, socially-negotiated judgment of which problems matter for the field
  → Schouten gives a proto-Kuhnian story of disciplinary tension and change.
  → Formal definitions settle consistency but not significance

Opening ceremony of the Oslo 1936 congress (Oslo National Library; Hollings & Siegmund-Schultze 2020)



COMPTES RENDUS
DU
CONGRÈS INTERNATIONAL
DES MATHÉMATICIENS
OSLO 1936

Tome I

Procès-Verbaux
et
Conférences Générales

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What does it really matter whether an investigation is called ‘geometrical’ or ‘analytical,’ when what is essential is simply that the problem is important and the treatment is good? And it is precisely in this evaluation of problems, in the distinction between what at a given moment is important and what is **not** or **not yet** important, that the emotional element [het emotioneele element] plays **the** role.

Schouten 1939, p. 11, original emphasis)

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